Let $S = \{(x, y) : x, y$ are rational, $x < y, x^x = y^y\}$. Note that $(1/4, 1/2)$ is in S. Find an accumulation point of S or prove that S has no accumulation points.

704. *Proposed by Roger B. Nelsen, Lewis & Clark College, Portland, OR*

We say that a continuous random variable is *symmetric about zero* if the density function of the random variable is an even function. Let X and Y be identically distributed continuous random variables. Prove or disprove:

(a) The difference $X - Y$ is symmetric about zero.

(b) If X and Y are symmetric about zero, then so is the sum $X + Y$.

Do your answers in (a) or (b) change if X and Y are also assumed to be independent?

705. *Proposed by Ayoub B. Ayoub, Pennsylvania State University, Abington College, Abington, PA*

If $0 < a \le b$, prove that

$$a \le \frac{ab\,(a+b)}{a^2+b^2} \le \sqrt[a+b]{a^b b^a} \le \frac{2ab}{a+b} \le \sqrt{ab} \le \frac{a+b}{2} \le \sqrt[a+b]{a^a b^b} \le \frac{a^2+b^2}{a+b} \le b.$$

SOLUTIONS

A Viewing Window for Limaçons

676. *Proposed by Rick Mabry, LSU–Shreveport and Paul Deiermann, Lindenwood University*

Let $r(\theta) = 1 + b\cos(\theta)$, where $0 < b \le 1$, describe a limaçon in polar coordinates. Determine the smallest rectangle of the form $[x_1, x_2] \times [y_1, y_2]$ that contains all these graphs. (This rectangle could be used as a fixed viewing window that contains the graphs of each of the limaçons.)

Solution by Jack V. Wales, Jr., The Thacher School, Ojai, CA

We seek the supremum and infimum of $x = r\cos(\theta) = \cos(\theta) + b\cos^2(\theta)$ and $y = r\sin(\theta) = \sin(\theta) + \frac{b}{2}\sin(2\theta)$ over $0 < b \le 1$, $0 \le \theta \le 2\pi$. Since both x and y are continuous functions of b at $b = 0$ we can extend the domain to include $b = 0$.

Since $\cos^2(\theta) \ge 0$, it is clear that $-1 \le \cos(\theta) \le x \le \cos(\theta) + \cos^2(\theta) \le 2$. For $b = 1$ and $\theta = 0$, $x = 2$ and for $b = 0$ and $\theta = \pi$, $x = -1$. Thus $[x_1, x_2] = [-1, 2]$.

Since the graph of the limaçon is symmetric about the x axis, $y_1 = -y_2$. For $0 \le \theta \le \frac{\pi}{2}$, $\sin(2\theta)$ is positive and for $\frac{\pi}{2} \le \theta \le \pi$, $\sin(2\theta)$ is negative. Thus for each θ in the latter interval, y attains a maximum when $b = 0$, and for each θ in the former interval, y attains a maximum when $b = 1$. On $\frac{\pi}{2} \le \theta \le \pi$ with $b = 0$, y attains a maximum of 1 at $\theta = \frac{\pi}{2}$. Thus the maximum value of y will occur on $0 \le \theta \le \frac{\pi}{2}$ with $b = 1$. Standard calculus techniques reveal that this happens at $\theta = \frac{\pi}{3}$. Therefore, $[y_1, y_2] = [-\frac{3\sqrt{3}}{4}, \frac{3\sqrt{3}}{4}]$

Also solved by MICHEL BATAILLE, Rouen, France; BEN B. BOWEN, Vallejo, CA; JEREMY CASE, Taylor U.; ROBERT D. CRISE, Jr.; RICHARD DAQUILA, Muskingum C.; JAMES DEUMMEL, Bellingham, WA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, CT; GREG DRESDEN, Washington and Lee U.; BILL DUNN, III, Montgomery C.; BILL GERSON, Prince Georges C. C.; JOHN GRAHAM, Penn State Wilkes-Barre; RICKY IKEDA, Leeward C. C.; PETER M. JARVIS, Georgia C. & State U.; KIM McINTURFF, Santa Barbara, CA; THOMAS

J. OSLER, Rowan U.; WILLIAM SEAMAN, Albright C.; R. S. TIBERIO, Natick, MA; SAMUEL A. TRUITT, Jr., Middle Tennessee State U.; THOMAS VANDEN EYNDEN, Thomas More C.; DOUG WILCOCK, Cape Cod Academy, MA; LI ZHOU, Polk C. C.; and the proposer.

An Inverse Function

677. *Proposed by Geoffrey A. Kandall. Hamden, CT*

The function $f : (0, \infty) \to (-\infty, \infty)$ defined by $f(t) = \frac{\sinh(2t)}{2\sinh(t)} - \coth(t)$ is increasing and onto. Derive an explicit formula, that involves only algebraic functions and natural logarithms, for the inverse function f^{-1}.

Solution by M. Reza Akhlaghi, Prestonsburg Community College, Prestonsburg, KY

The function f satisfies

$$y = f(t) = \frac{(1 + e^{2t})(e^{2t} - 2e^t - 1)}{2e^t(e^{2t} - 1)}$$

with $f(\ln(1 + \sqrt{2})) = 0$. Let $u = e^t$. Solving for u in terms of y, we are led to

$$u^4 - 2(y + 1)u^3 + 2(y - 1)u - 1 = 0.$$

This equation factors:

$$\left(u^2 - (y + 1)u - y - \sqrt{y^2 + 1}(u + 1)\right)\left(u^2 - (y + 1)u - y + \sqrt{y^2 + 1}(u + 1)\right) = 0.$$

The fact that $y = 0$ when $u = 1 + \sqrt{2}$ shows that only the left factor will yield a solution; using the quadratic formula it also shows that

$$u = \frac{1}{2}\left(y + 1 + \sqrt{y^2 + 1} + \sqrt{\left(y + 1 + \sqrt{y^2 + 1}\right)^2 + 4\left(y + \sqrt{y^2 + 1}\right)}\right)$$

is the only acceptable solution. The desired function is $t = f^{-1}(y) = \ln(u)$.

Also solved by MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian C.; JOSEPH COSTER, Macomb, IL; DANIELE DONINI, Bertinoro, Italy; JAMES DUEMMEL, Bellingham, WA; BILL DUNN, III, Montgomery C.; FLORIDA GULF COAST PROBLEM GROUP, Florida Gulf Coast U; JOHN GRAHAM, Penn State Wilkes-Barre; MURRAY S. KLAMKIN, U. of Alberta; HARRIS KWONG, SUNY C. at Fredonia; KIM McINTURFF, Santa Barbara, CA; STEPHEN NOLTIE, Ohio U.- Lancaster; WILLIAM SEAMAN, Albright C.; CORNELIUS STALLMAN and GERALD THOMPSON, Augusta State U.; SAMUEL A. TRUITT, JR., Middle Tennessee State U.; OMER YAYENIE and MOHAMUD MOHAMMED, Temple U.; LI ZHOU, Polk C.C.; and the proposer.

A Double Sum

678. *Proposed by David Atkinson, Olivet Nazarene University, Kankakee, IL*

For $n = 0, 1, \ldots$, find the value of the double sum $\sum_{i=0}^{n} \sum_{j=0}^{n-i} \frac{(-1)^j}{i!j!}$ as a function of n.